

An interesting and seminal work on various phenomena in fluid mechanics

Alan N. Other^{1†}, H.-C. Smith¹
and J.Q. Public²

¹Department of Chemical Engineering, University of America, Somewhere, IN 12345, USA
email:

²Department of Aerospace and Mechanical Engineering, University of Camford,
Academic Street, Camford, CF3 5QL, UK
email:

Abstract. Using Stokes flow between eccentric counter-rotating cylinders as a prototype for bounded nearly parallel lubrication flow, we investigate the effect of a slender recirculation region within the flow field on cross-stream heat or mass transport in the important limit of high Péclet number Pe where the ‘enhancement’ over pure conduction heat transfer without recirculation is most pronounced. The steady enhancement is estimated with a matched asymptotic expansion to resolve the diffusive boundary layers at the separatrices which bound the recirculation region. The enhancement over pure conduction is shown to vary as $\epsilon^{1/2}$ at infinite Pe , where $\epsilon^{1/2}$ is the characteristic width of the recirculation region. The enhancement decays from this asymptote as $Pe^{-1/2}$.

Keywords. Keyword1, keyword2, keyword3, etc.

1. Introduction

The use of integral equations to solve ‘exterior’ problems in linear acoustics, i.e. to solve the Helmholtz equation $(\nabla^2 + k^2)\phi = 0$ outside a surface S given that ϕ satisfies certain boundary conditions on S , is very common. A good description is provided by Martin (1980). Integral equations have also been used to solve the two-dimensional Helmholtz equation that arises in water-wave problems where there is a constant depth variation. The problem of wave oscillations in arbitrarily shaped harbours using such techniques has been examined (see for example Hwang & Tuck 1970; Lee 1971; Figer, Najarro, Gilmore, *et al.* (2002)).

In a recent paper Linton & Evans (1992) have shown how radiation and scattering problems for vertical circular cylinders placed on the centreline of a channel of finite water depth can be solved efficiently using the multipole method devised originally by Ursell (1950). This method was also used by Callan, Linton & Evans (1991) to prove the existence of trapped modes in the vicinity of such a cylinder at a discrete wavenumber $k < \pi/2d$ where $2d$ is the channel width.

Many water-wave/body interaction problems in which the body is a vertical cylinder with constant cross-section can be simplified by factoring out the depth dependence. Thus if the boundary conditions are homogeneous we can write the velocity potential $\phi(x, y, z, t) = \text{Re}\{\phi(x, y) \cosh k(z + h)e^{-i\omega t}\}$, where the (x, y) -plane corresponds to the undisturbed free surface and z is measured vertically upwards with $z = -h$ the bottom of the channel.

† Present address: Fluid Mech Inc., 24 The Street, Lagos, Nigeria.

Subsequently Callan *et al.* (1991) proved the existence of, and computed the wavenumbers for, the circular cross-section case. It should be noted however that experimental evidence for acoustic resonances in the case of the circular cylinder is given by Bearman & Graham (1980, pp. 231–232).

Koch (1983) provided a theory for determining the trapped-mode frequencies for the thin plate, based on a modification of the Wiener–Hopf technique. Further interesting results can be found in Williams (1964) and Dennis (1985).

The use of channel Green’s functions allows the far-field behaviour to be computed in an extremely simple manner, whilst the integral equation constructed in §3 enables the trapped modes to be computed in §4 and the scattering of an incident plane wave to be solved in §5. Appendix A contains comparisons with experiments. The Galactic Center region proves to be an area very rich in WR stars. The VIIth Catalogue lists within 50 pc from the Galactic Center 15 WNL and 11 WCL stars, at near-IR wavelengths discovered by Krabbe, Genzel, Eckart, *et al.* (1995) in the Galactic Center Cluster.

2. Green’s functions

2.1. Construction of equations

We are concerned with problems for which the solution, ϕ , is either symmetric or anti-symmetric about the centreline of the waveguide, $y = 0$. The first step is the construction of a symmetric and an antisymmetric Green’s function, $G_s(P, Q)$ and $G_a(P, Q)$. Thus we require

$$(\nabla^2 + k^2)G_s = (\nabla^2 + k^2)G_a = 0 \quad (2.1)$$

in the fluid, where ∇ is a gradient operator,

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla^2 P = \nabla \cdot (\mathbf{v} \times \mathbf{w}).$$

In (2.1)

$$G_s, G_a \sim 1/(2\pi) \ln r \quad \text{as} \quad r \equiv |P - Q| \rightarrow 0, \quad (2.2)$$

$$\frac{\partial G_s}{\partial y} = \frac{\partial G_a}{\partial y} = 0 \quad \text{on} \quad y = d, \quad (2.3)$$

$$\frac{\partial G_s}{\partial y} = 0 \quad \text{on} \quad y = 0, \quad (2.4)$$

$$G_a = 0 \quad \text{on} \quad y = 0, \quad (2.5)$$

and we require G_s and G_a to behave like outgoing waves as $|x| \rightarrow \infty$.

One way of constructing G_s or G_a is to replace (2.1) and (2.2) by

$$(\nabla^2 + k^2)G_s = (\nabla^2 + k^2)G_a = -\delta(x - \xi)\delta(y - \eta) \quad (2.6)$$

and to assume initially that k has a positive imaginary part.

2.2. Further developments

Using results from Linton & Evans (1992) we see that this has the integral representation

$$-\frac{1}{2\pi} \int_0^\infty \gamma^{-1} [e^{-k\gamma|y-\eta|} + e^{-k\gamma(2d-y-\eta)}] \cos k(x - \xi)t \, dt, \quad 0 < y, \quad \eta < d, \quad (2.7)$$

where

$$\gamma(t) = \begin{cases} -i(1 - t^2)^{1/2}, & t \leq 1 \\ (t^2 - 1)^{1/2}, & t > 1. \end{cases}$$

In order to satisfy (2.4) we add to this the function

$$-\frac{1}{2\pi} \int_0^\infty B(t) \frac{\cosh k\gamma(d-y)}{\gamma \sinh k\gamma d} \cos k(x-\xi)t dt$$

which satisfies (2.1), (2.3) and obtain

$$B(t) = 2e^{-k\gamma d} \cosh k\gamma(d-\eta). \quad (2.8)$$

Thus the function

$$G = -\frac{1}{4}i(H_0(kr) + H_0(kr_1)) - \frac{1}{\pi} \int_0^\infty \frac{e^{-k\gamma d}}{\gamma \sinh k\gamma d} \cosh k\gamma(d-y) \cosh k\gamma(d-\eta) \quad (2.9)$$

satisfies (2.1)–(2.4). By writing this function as a single integral which is even in γ , it follows that G is real. Similar ideas have been developed in a variety of ways (Keller 1977; Rogallo 1981; van Wijngaarden 1968).

3. The trapped-mode problem

The unit normal from D to ∂D is $\mathbf{n}_q = (-y'(\theta), x'(\theta))/w(\theta)$. Now $G_a = \frac{1}{4}Y_0(kr) + \widetilde{G}_a$ where $r = \{[x(\theta) - x(\psi)]^2 + [y(\theta) - y(\psi)]^2\}^{1/2}$ and \widetilde{G}_a is regular as $kr \rightarrow 0$. In order to evaluate $\partial G_a(\theta, \theta)/\partial n_q$ we note that

$$\frac{\partial}{\partial n_q} \left(\frac{1}{4}Y_0(kr) \right) \sim \frac{\partial}{\partial n_q} \left(\frac{1}{2\pi} \ln r \right) = \frac{1}{2\pi r^2 w(\theta)} [x'(\theta)(y(\theta) - y(\psi)) - y'(\theta)(x(\theta) - x(\psi))]$$

as $kr \rightarrow 0$. Expanding $x(\psi)$ and $y(\psi)$ about the point $\psi = \theta$ then shows that

$$\begin{aligned} \frac{\partial}{\partial n_q} \left(\frac{1}{4}Y_0(kr) \right) &\sim \frac{1}{4\pi w^3(\theta)} [x''(\theta)y'(\theta) - y''(\theta)x'(\theta)] \\ &= \frac{1}{4\pi w^3(\theta)} [\rho'(\theta)\rho''(\theta) - \rho^2(\theta) - 2\rho'^2(\theta)] \quad \text{as } kr \rightarrow 0. \end{aligned} \quad (3.1)$$

3.1. Computation

For computational purposes we discretize (3.1) by dividing the interval $(0, \pi)$ into M segments. Thus we write

$$\frac{1}{2}\phi(\psi) = \frac{\pi}{M} \sum_{j=1}^M \phi(\theta_j) \frac{\partial}{\partial n_q} G_a(\psi, \theta_j) w(\theta_j), \quad 0 < \psi < \pi, \quad (3.2)$$

where $\theta_j = (j - \frac{1}{2})\pi/M$. Collocating at $\psi = \theta_i$ and writing $\phi_i = \phi(\theta_i)$ etc. gives

$$\frac{1}{2}\phi_i = \frac{\pi}{M} \sum_{j=1}^M \phi_j K_{ij}^a w_j, \quad i = 1, \dots, M, \quad (3.3)$$

where

$$K_{ij}^a = \begin{cases} \partial G_a(\theta_i, \theta_j)/\partial n_q, & i \neq j \\ \partial \widetilde{G}_a(\theta_i, \theta_i)/\partial n_q + [\rho'_i \rho''_i - \rho_i^2 - 2\rho_i'^2]/4\pi w_i^3, & i = j. \end{cases} \quad (3.4)$$

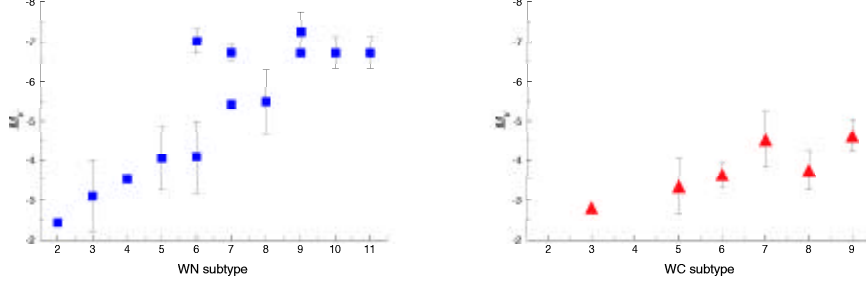
For a trapped mode, therefore, we require the determinant of the $M \times M$ matrix whose elements are

$$\delta_{ij} - \frac{2\pi}{M} K_{ij}^a w_j,$$

to be zero. Table 1 shows a comparison of results obtained from this method using two

Table 1. Values of kd at which trapped modes occur when $\rho(\theta) = a$

a/d	$M = 4$	$M = 8$	Callan <i>et al.</i>
0.1	1.56905	1.56	1.56904
0.3	1.50484	1.504	1.50484
0.55	1.39128	1.391	1.39131
0.7	1.32281	10.322	1.32288
0.913	1.34479	100.351	1.35185

**Figure 1.** Trapped-mode wavenumbers, kd , plotted against a/d for three ellipses: $---$, $b/a = 0.5$; $—$, $b/a = 1$; $- \cdot -$, $b/a = 1.5$.

different truncation parameters with accurate values obtained using the method of Callan *et al.* (1991).

An example of the results that are obtained from our method is given in figure 1. Figure 2(a,b) shows shaded contour plots of ϕ for these modes, normalized so that the maximum value of ϕ on the body is 1. Symmetric (figure 2a) modes are shown, while the antisymmetric ones appear in figure 2(b).

3.2. Basic properties

Let

$$\rho_l = \lim_{\zeta \rightarrow Z_l^-(x)} \rho(x, \zeta), \quad \rho_u = \lim_{\zeta \rightarrow Z_u^+(x)} \rho(x, \zeta) \quad (3.5a, b)$$

be the fluid densities immediately below and above the cat's-eyes. Finally let ρ_0 and N_0 be the constant values of the density and the vorticity inside the cat's-eyes, so that

$$(\rho(x, \zeta), \phi_{\zeta\zeta}(x, \zeta)) = (\rho_0, N_0) \quad \text{for} \quad Z_l(x) < \zeta < Z_u(x). \quad (3.6)$$

The Reynolds number Re is defined by $u_\tau H/\nu$ (ν is the kinematic viscosity), the length given in wall units is denoted by $(\cdot)_+$, and the Prandtl number Pr is set equal to 0.7. In (2.1) and (2.2), τ_{ij} and τ_j^θ are

$$\tau_{ij} = (\overline{u_i u_j} - \overline{u_i} \overline{u_j}) + (\overline{u_i u_j^{SGS}} + \overline{u_i^{SGS} u_j}) + \overline{u_i^{SGS} u_j^{SGS}}, \quad (3.7a)$$

$$\tau_j^\theta = (\overline{u_j \theta} - \overline{u_j} \overline{\theta}) + (\overline{u_j \theta^{SGS}} + \overline{u_j^{SGS} \theta}) + \overline{u_j^{SGS} \theta^{SGS}}. \quad (3.7b)$$

3.2.1. Calculation of the terms

The first terms in the right-hand side of (3.5a) and (3.5b) are the Leonard terms explicitly calculated by applying the Gaussian filter in the x - and z -directions in the Fourier space.

The interface boundary conditions given by (2.1) and (2.2), which relate the displacement and stress state of the wall at the mean interface to the disturbance quantities of the

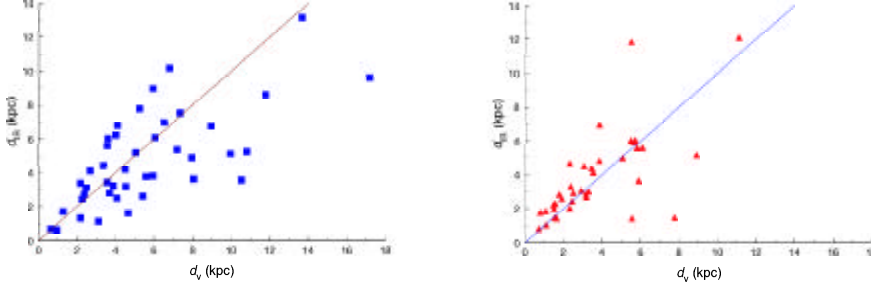


Figure 2. Shaded contour plots of the potential ϕ for the two trapped modes that exist for an ellipse with $a/d = 1.5$, $b/d = 0.75$. (a) Symmetric about $x = 0$, $kd = 0.96$; (b) antisymmetric about $x = 0$, $kd = 1.398$.

flow, can also be reformulated in terms of the transformed quantities. The transformed boundary conditions are summarized below in a matrix form that is convenient for the subsequent development of the theory:

$$Q_C = \begin{bmatrix} -\omega^{-2}V'_w & -(\alpha^t\omega)^{-1} & 0 & 0 & 0 \\ \frac{\beta}{\alpha\omega^2}V'_w & 0 & 0 & 0 & i\omega^{-1} \\ i\omega^{-1} & 0 & 0 & 0 & 0 \\ iR_\delta^{-1}(\alpha^t + \omega^{-1}V''_w) & 0 & -(i\alpha^t R_\delta)^{-1} & 0 & 0 \\ \frac{i\beta}{\alpha\omega}R_\delta^{-1}V''_w & 0 & 0 & 0 & 0 \\ (i\alpha^t)^{-1}V'_w & (3R_\delta^{-1} + c^t(i\alpha^t)^{-1}) & 0 & -(\alpha^t)^{-2}R_\delta^{-1} & 0 \end{bmatrix}. \quad (3.8)$$

\mathbf{S}^t is termed the displacement-stress vector and Q_C the flow-wall coupling matrix. Subscript w in (3.8) denotes evaluation of the terms at the mean interface. It is noted that $V''_w = 0$ for the Blasius mean flow.

3.2.2. Wave propagation in anisotropic compliant layers

From (2.1), the fundamental wave solutions to (3.1) and (3.2) for a uniformly thick homogeneous layer in the transformed variables has the form of

$$\boldsymbol{\eta}^t = \hat{\boldsymbol{\eta}}^t \exp[i(\alpha^t x_1^t - \omega t)], \quad (3.9)$$

where $\hat{\boldsymbol{\eta}}^t = \mathbf{b} \exp(i\gamma x_3^t)$. For a non-trivial wave, the substitution of (1.7) into (1.3) and (1.4) yields the following determinantal equation:

$$\text{Det}[\rho\omega^2\delta_{ps} - C_{pqrs}^t k_q^t k_r^t] = 0, \quad (3.10)$$

where the wavenumbers $\langle k_1^t, k_2^t, k_3^t \rangle = \langle \alpha^t, 0, \gamma \rangle$ and δ_{ps} is the Kronecker delta.

4. Torus translating along an axis of symmetry

Consider a torus with axes a, b (see figure 2), moving along the z -axis. Symmetry considerations imply the following form for the stress function, given in body coordinates:

$$\mathbf{f}(\theta, \psi) = (g(\psi) \cos \theta, g(\psi) \sin \theta, f(\psi)). \quad (4.1)$$

Because of symmetry, one can integrate analytically in the θ -direction obtaining a pair

of equations for the coefficients f, g in (4.1),

$$f(\psi_1) = \frac{3b}{\pi[2(a+b\cos\psi_1)]^{3/2}} \int_0^{2\pi} \frac{(\sin\psi_1 - \sin\psi)(a+b\cos\psi)^{1/2}}{[1-\cos(\psi_1-\psi)](2+\alpha)^{1/2}} d\psi, \quad (4.2)$$

$$\begin{aligned} g(\psi_1) = & \frac{3}{\pi[2(a+b\cos\psi_1)]^{3/2}} \int_0^{2\pi} \left(\frac{a+b\cos\psi}{2+\alpha} \right)^{1/2} \left\{ f(\psi)[(\cos\psi_1 - b\beta_1)S + \beta_1 P] \right. \\ & \times \frac{\sin\psi_1 - \sin\psi}{1-\cos(\psi_1-\psi)} + g(\psi) \left[\left(2+\alpha - \frac{(\sin\psi_1 - \sin\psi)^2}{1-\cos(\psi-\psi_1)} - b^2\gamma \right) S \right. \\ & \left. \left. + \left(b^2\cos\psi_1\gamma - \frac{a}{b}\alpha \right) F(\tfrac{1}{2}\pi, \delta) - (2+\alpha)\cos\psi_1 E(\tfrac{1}{2}\pi, \delta) \right] \right\} d\psi, \end{aligned} \quad (4.3)$$

$$\alpha = \alpha(\psi, \psi_1) = \frac{b^2[1-\cos(\psi-\psi_1)]}{(a+b\cos\psi)(a+b\cos\psi_1)}; \quad \beta = \beta(\psi, \psi_1) = \frac{1-\cos(\psi-\psi_1)}{a+b\cos\psi}. \quad (4.4)$$

5. Conclusions

We have shown how integral equations can be used to solve a particular class of problems concerning obstacles in waveguides, namely the Neumann problem for bodies symmetric about the centreline of a channel, and two such problems were considered in detail.

Appendix A. Boundary conditions

It is convenient for numerical purposes to change the independent variable in (4.1) to $z = y/\tilde{v}_T^{1/2}$ and to introduce the dependent variable $H(z) = (f - \tilde{y})/\tilde{v}_T^{1/2}$. Equation (4.1) then becomes

$$(1-\beta)(H+z)H'' - (2+H')H' = H'''. \quad (A 1)$$

Boundary conditions to (4.3) follow from (4.2) and the definition of H :

$$\left. \begin{aligned} H(0) &= \frac{\epsilon\overline{C}_v}{\tilde{v}_T^{1/2}(1-\beta)}; & H'(0) &= -1 + \epsilon^{2/3}\overline{C}_u + \epsilon\hat{C}'_u; \\ H''(0) &= \frac{\epsilon u_*^2}{\tilde{v}_T^{1/2}u_P^2}; & H'(\infty) &= 0. \end{aligned} \right\} \quad (A 2)$$

Appendix B

A simple sufficient condition for the method of separation of variables to hold for the convection problem is derived. This criterion is then shown to be satisfied for the ansatz described by (3.27), thus justifying the approach used in § 3. The basic ingredient of our argument is contained in the following estimate for a Rayleigh–Ritz ratio:

LEMMA 1. *Let $f(z)$ be a trial function defined on $[0, 1]$. Let Λ_1 denote the ground-state eigenvalue for $-d^2g/dz^2 = \Lambda g$, where g must satisfy $\pm dg/dz + \alpha g = 0$ at $z = 0, 1$ for*

some non-negative constant α . Then for any f that is not identically zero we have

$$\frac{\alpha(f^2(0) + f^2(1)) + \int_0^1 \left(\frac{df}{dz}\right)^2 dz}{\int_0^1 f^2 dz} \geq \Lambda_1 \geq \left(\frac{-\alpha + (\alpha^2 + 8\pi^2\alpha)^{1/2}}{4\pi}\right)^2. \quad (\text{B } 1)$$

Before proving it, we note that the first inequality is the standard variational characterization for the eigenvalue Λ_1 .

COROLLARY 1. Any non-zero trial function f which satisfies the boundary condition $f(0) = f(1) = 0$ always satisfies

$$\int_0^1 \left(\frac{df}{dz}\right)^2 dz. \quad (\text{B } 2)$$

Acknowledgements

We would like to acknowledge the useful comments of a referee concerning the solution procedure used in § 5. A. N. O. is supported by SERC under grant number GR/F/12345.

References

- Bearman, P.W. & Graham, J.M.R. 1980, *J. Fluid Mech.* 99, 225
 Callan, M., Linton, C.M. & Evans D.V. 1991, *J. Fluid Mech.* 229, 51
 Dennis, S.C.R. 1985, in: X. Soubbaramayer & J.P. Boujot (eds.), *Ninth Intl. Conf. on Numerical Methods in Fluid Dynamics*, Lecture Notes in Physics (Heidelberg: Springer), vol. 218, p. 23
 Figer, D.F., Najarro, F., Gilmore, D., Morris, M., Kim, S.S., Serabyn, E., McLean, I.S., Gilbert, A.M., Graham, J.R., Larkin, J.E., Levenson, N.A., Teplitz, H.I. 2002, *ApJ* 581, 258
 Hwang, L.-S. & Tuck, E. O. 1970, *J. Fluid Mech.* 42, 447
 Keller, H.B. 1977 in: P.H. Rabinovich (ed.), *Applications of Bifurcation Theory* (Academic), p. 359
 Koch, W. 1983, *J. Sound Vib.* 88, 233
 Krabbe, A., Genzel, R., Eckart, A., Najarro, F., Lutz, D., Cameron, M., Kroker, H., Tacconi-Garman, L. E., Thatte, N., Weitzel, L., Drapatz, S., Geballe, T., Sternberg, A., Kudritzki, R.-P. 1995, *ApJ* (Letters) 447, L95
 Lee, J.-J. 1971, *J. Fluid Mech.* 45, 375
 Linton, C.M. & Evans, D.V. 1992, *Phil. Trans. R. Soc. Lond.* A338, 325
 Martin, P.A. 1980, *Q. J. Mech. Appl. Maths* 33, 385
 Rogallo, R.S. 1981, *NASA Tech. Mem.* 81835.
 Ursell, F. 1950, *Proc. Camb. Phil. Soc.* 46, 141
 van Wijngaarden, L. 1968, *J. Engng Maths* 2, 225
 Williams, J.A. 1964, PhD thesis, University of California, Berkeley.

Discussion

MASSEY: Im wondering if you have considered the expected intrinsic dispersion in absolute magnitude of WRs – if you consider the (large) mass range that becomes an early WN or late WC according to the evolutionary models, wouldnt you expect a large dispersion in M_v ?

VAN DER HUUCHT: Indeed, we will be always left with some intrinsic scatter in M_v due to mass differences within the same spectral subtype. But in my opinion, the current large

dispersion is for a large fraction due to uncertainties of the adopted distances of open clusters and OB associations.

WALBORN: I think that the scatter in WNL absolute magnitudes is dominated by intrinsic spread rather than errors. In the LMC, one finds a range of 5 to nearly 8. This in turn likely reflects different formation channels: mass-transfer binaries, post-RSG, and extremely massive stars in giant H II regions.

VAN DER HUHT: As said above, there is likely to be intrinsic scatter. But, I wonder whether a scatter of 3 magnitudes perhaps reflects undetected multiplicity.

MAÍZ-APELLÁNIZ: I could not agree more with your comment on the need for an updated catalogue of O-type stars (as a follow up of that of Garmany *et al.* 1982). We are currently working on precisely that (see our poster, these Proceedings) and we will soon make it available.

VAN DER HUHT: Wonderful.

KOENIGSBERGER: Is the ratio WR/O-stars in clusters similar or different from this ratio for the field stars?

VAN DER HUHT: I think it is different because incompleteness among field stars is even larger than that among cluster stars. But perhaps it should also be different because WR stars are older and could have drifted away from clusters, more than O-type stars.

GIES: How many of the WR stars in your catalogue might be low mass objects?

WALBORN: Comment: PN central stars in the WR sample would be only [WC].

VAN DER HUHT: Among the WR stars in our VIIth Catalogue we doubt only one: WR109 (V617 Sgr), which is a peculiar object (not even a [WR] central star of a PN). All other stars in our catalogue are true massive Population I WR stars, and properly classified as such. We have not listed known Population II [WC] objects, as we did separately in our VIth Catalogue (van der Hucht *et al.* 1981). [WN] objects are not known to exist, see the comment by Nolan.

ZINNECKER: Are all Galactic WR stars in open clusters and OB association or are there many WR stars in the field?

VAN DER HUHT: See the VIIth WR catalogue (van der Hucht 2001): of the listed 227 Galactic WR stars, only 53 are in open clusters and OB associations, or believed to be. The other 184 are supposedly field stars.