Secular Dynamics of Binaries in Stellar Clusters

Chris Hamilton

with Roman Rafikov | IAUGA, Busan | August 2022
§1. Introduction. Two body problem, three body problem, Lidov-Kozai mechanism.

§2. Motivation. What do black hole binaries and Oort cloud comets have in common?

§3. Theory. General secular theory of dynamical evolution of binaries in axisymmetric potentials.

§1. Introduction

Two body problem,

three body problem;

Lidov-Kozai mechanism.
Two-body problem

• Two point masses $m_1, m_2$ in Newtonian gravity.

• Exactly solvable: ellipse with semi major axis $a$, eccentricity $e$

\[ p = a(1-e) \]

\[ H_0 = -G\left(m_1 + m_2\right)/2a \]
Three-body problem

• In general, chaotic & unstable!
Three-body problem

- In general, chaotic & unstable!
Three-body problem

- In general, chaotic & unstable!

Hut & Bahcall (1983)
The *hierarchical* three-body problem

- \( a_{\text{out}} \gg a \)
- Stable
- Can be treated perturbatively

\[
H = H_0 + H_1
\]

Two-body Hamiltonian perturbation by third body

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The hierarchical three-body problem

Secular theory for $e(t), i(t), \omega(t), \Omega(t)$ in 2 easy steps:

**Step 1**
Time-average over **inner binary’s** orbital period (`single averaging`).

**Step 2**
Time-average over **outer body’s** orbital period (`double averaging`).
The hierarchical three-body problem

Key result: a new constant of motion

$$\sqrt{1 - e^2 \cos i}$$

so relative inclination of orbits can be exchanged for eccentricity!
The *hierarchical* three-body problem

**Key result:** a new constant of motion

\[ \sqrt{1 - e^2 \cos i} \]

so relative inclination of orbits can be exchanged for eccentricity!

If \( i_0 \approx 40 \) degrees, e oscillations can be very large!

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The hierarchical three-body problem

Key result: a new constant of motion

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so inclination can be exchanged for eccentricity!

If \( i_0 \approx 40 \) degrees, \( e \) oscillations can be very large!
This *Lidov-Kozai* mechanism is a way to **excite eccentricity**, applicable to...

- black hole triples
- stellar triples
- circumbinary planets
- binaries near SMBHs...

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**Martin & Triaud 2015**

(a) classic Kozai-Lidov
(b) triple stars
(c) multi-planet
(d) circumbinary
Okay... So what?

- High-e oscillations reduce the closest-approach distance $p=a(1-e)$ of a binary.

- When $p$ is small, a BH-BH binary’s semimajor axis $a$ shrinks via GW emission at each pericentre passage:

- Many papers have suggested Lidov-Kozai-induced mergers of BHs, NSs can explain LIGO events.

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Lidov-Kozai dynamics

4 key quantities:

- Semimajor axis $a$
- Eccentricity $e$
- Inclination $i$
- Argument of pericentre $\omega$

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Lidov-Kozai dynamics

4 key quantities:

- Semimajor axis $a$
- Eccentricity $e$
- Inclination $i$
- Argument of pericentre $\omega$
Lidov-Kozai dynamics

3 constants of motion:

Semimajor axis $a$

‘Angular momentum’

$\Theta \equiv (1 - e^2) \cos^2 i$

‘Energy’

$H_{1}^{LK}(e, i, \omega) = (2 + 3e^2) (1 - 3 \cos^2 i) - 15e^2 \sin^2 i \cos 2\omega$. 

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Lidov-Kozai dynamics

So the system is confined to a 1D contour in 2D phase space:

\[ H_{1}^{LK} = (2 + 3e^2)(1 - 3\cos^2i) - 15e^2 \sin^2 i \cos 2\omega. \]
§2. Motivation

What do black-hole binaries and Oort cloud comets have in common?
I’ve talked as if the binary’s barycentre sits still and the outer perturber whizzes round it…

…but we could equivalently say that a fixed perturber is orbited by the binary.
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Oort Cloud comets

The comet-sun ‘binary’ orbits the Galaxy and performs eccentricity oscillations.

Heisler & Tremaine (1986)
Doubly-averaged secular theory gives 3 constants of motion:

- **Semimajor axis**  
  \[ a \]

- **‘Angular momentum’**  
  \[ \Theta \equiv (1 - e^2) \cos^2 i \]

- **‘Energy’**  
  \[ H^{HT}_1(e, i, \omega) \]

\[
H^{HT}_1 = (2 + 3e^2) \sin^2 i - 5e^2 \sin^2 i \cos 2\omega.
\]
Lidov-Kozai (LK) and Heisler-Tremaine (HT) are very different problems, but qualitatively give very similar results…

\[ H_1^{LK} = (2 + 3e^2) (1 - 3 \cos^2 i) - 15e^2 \sin^2 i \cos 2\omega. \]

\[ H_1^{HT} = (2 + 3e^2) \sin^2 i - 5e^2 \sin^2 i \cos 2\omega. \]
§3. Theory

General secular theory

of dynamical evolution of binaries

in axisymmetric potentials.

Hamilton (2021, PhD Thesis)
Take any binary moving on an orbit $\mathbf{R}_g(t)$ in an arbitrary smooth, axisymmetric potential $\Phi$ (the `cluster').

Then the binary dynamics are described by a Hamiltonian:

$$H = H_0 + H_1$$

where $H_1$ is the perturbation due to the cluster.
• Expand $H_1$ for $|R_g| \gg a$:

• Average $H_1$ over inner binary’s elliptical motion (‘single averaging’);

• Average $H_1$ again over many orbits of the binary around the cluster (‘double averaging’).
Double-averaging

The binary’s barycentre gradually fills an axisymmetric torus with symmetry axis $Z$.

Second time average = weighted volume average over the filled torus.

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Double-averaging

The binary’s barycentre gradually fills an axisymmetric torus with symmetry axis Z.

Second time average = weighted volume average over the the filled torus.

https://www.ifa.hawaii.edu/~barnes/saas-fee/chapter7.html

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Doubly-averaged secular theory gives 3 constants of motion:

<table>
<thead>
<tr>
<th>Semimajor axis</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$\Theta \equiv (1 - e^2) \cos^2 i$</td>
<td>$H_1(e, i, \omega, \Gamma)$</td>
</tr>
</tbody>
</table>

$$H_1 = (2 + 3e^2) \left(1 - 3\Gamma \cos^2 i\right) - 15\Gamma e^2 \sin^2 i \cos 2\omega.$$ 

where $\Gamma$ depends on
- the background potential $\phi$
- the orbit of the binary’s barycentre within the potential.

Easily recover LK and HT:

- $\Gamma = 1$ $\leftrightarrow$ test-particle quadrupole Lidov-Kozai
- $\Gamma = 1/3$ $\leftrightarrow$ Heisler & Tremaine (binary in disk)
When $\Gamma > 1/5$, very similar to Lidov-Kozai ($\Gamma = 1$).

...but there are bifurcations when $\Gamma = 1/5$, 0, and -1/5...

\[ H_1 = (2 + 3e^2) \left(1 - 3\Gamma \cos^2 i\right) - 15\Gamma e^2 \sin^2 i \cos 2\omega. \]
So we’ve learned that:

‘Lidov-Kozai’-type eccentricity oscillations are in fact very general: e.g.

(a) the test particle hierarchical three-body problem,
   \textit{Lidov (1962), Kozai (1962)}

(b) Oort comets torqued by the Galactic tide,
   \textit{Heisler & Tremaine (1986)}

(c) stars in nuclear clusters orbiting a SMBH,
   \textit{Isawa & Seto (2016, 2017)}

and several others, all arise as special cases of the
\textit{general secular theory (Hamilton & Rafikov 2019a, 2019b, 2021)}.
§4. Application

Compact object binary mergers

in stellar clusters.

Hamilton & Rafikov
Isolated mergers

Merger time of isolated, circular binary $\propto a_0^4$.

A black hole binary ($m_1 = m_2 = 30M_\odot$) would only merge in the age of the Universe if $a_0 \lesssim 0.2$ AU.
Cluster tide-driven mergers

30 AU black hole binary merging after 850 Myr

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Cluster tide-driven mergers

Estimate of merger time:

\[ T_m \approx \frac{3c^5a_0^4}{85G^3(m_1 + m_2)m_1m_2} (1 - \tilde{c}_{\text{max}}^2)^3. \]

\[ \approx 1 \text{ Gyr} \times \left( \frac{m}{10M_\odot} \right)^{-3} \left( \frac{a_0}{10\text{AU}} \right)^4 \left( \frac{1 - \tilde{c}_{\text{max}}}{0.001} \right)^3. \]
Compact Object Binary Mergers Driven By Cluster Tides: A New Channel for LIGO/Virgo Gravitational-wave Events

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- Focus on BH-BH, BH-NS and NS-NS binaries in \textit{spherical} globular and nuclear star clusters.

- Can calculate $T_m(\Gamma, e_0, i_0, \ldots)$ (semi-)analytically!

- Perform Monte Carlo population synthesis calculation.

\implies get merger rates to compare to LIGO/Virgo.
Rates are quite low compared to LIGO/Virgo OR2...

...but several effects yet to be accounted for (e.g. stellar scattering, non-sphericity, SMBHs...)

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Bub & Petrovich 2019
Summary

- Binaries are **not isolated systems**: they respond to their environment.
- ‘Lidov-Kozai oscillations’ in $e$ are actually a very generic phenomenon.
- **Rich dynamics**, particularly if including GR/GWs. Subtle interplay between phase space structure and dynamical evolution.
- Tide-driven excitation mechanism can explain (some) LIGO/Virgo merger events. But some observables are in DF, not time series.

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Applying the same idea to develop kinetic theory of wide binaries in the Milky Way, to explain GAIA eccentricities!

\[ \frac{\partial f}{\partial t} + \{f, H\} = 0 \]

On the Phase-mixed Eccentricity and Inclination Distributions of Wide Binaries in the Galaxy

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