Effects of the mass redistribution on the rotation of the Earth

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- This talk is based on the papers:
  - ***Precession of the non-rigid Earth: Effect of the mass redistribution*** by Baenas, Escapa, & Ferrándiz (A&A 626, A58, 2019, https://doi.org/10.1051/0004-6361/201935472)
  - ***Nutation of the non-rigid Earth: Effect of the mass redistribution*** by Baenas, Escapa, & Ferrándiz (A&A 643, A159, 2020, https://doi.org/10.1051/0004-6361/202038946)
  - ***Secular changes in length of day: Effect of the mass redistribution*** by Baenas, Escapa, & Ferrándiz (A&A, 648, A89, 2021, https://doi.org/10.1051/0004-6361/202140356)
- We refer the reader to those publications in order to find the precise definition of the notations and symbols used in the next slides, as well as a comprehensive discussion on this topic

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Context and classical formulae

- The Earth as an extended body is affected, among others mechanisms, by the gravitational action that the Moon, the Sun, and the planets, i.e., the perturbers, exert on it
- Such interaction manifests in:
  - Changes in the Earth's figure axis (precession/nutation) and angular velocity vector (polar motion and length of day)
  - Tidal effects that the perturbers create on the non-rigid Earth, which is responsible of a redistribution of mass of tidal origin
- In a raw first approximation both aspects can be considered independent (e.g., a rigid model for Earth's figure axis changes), but in fact they are coupled in many complex ways
- In this talk, we will focus on the effects that the tidal redistribution of mass produce in the precession, the nutation, and the length of day (LOD) variations

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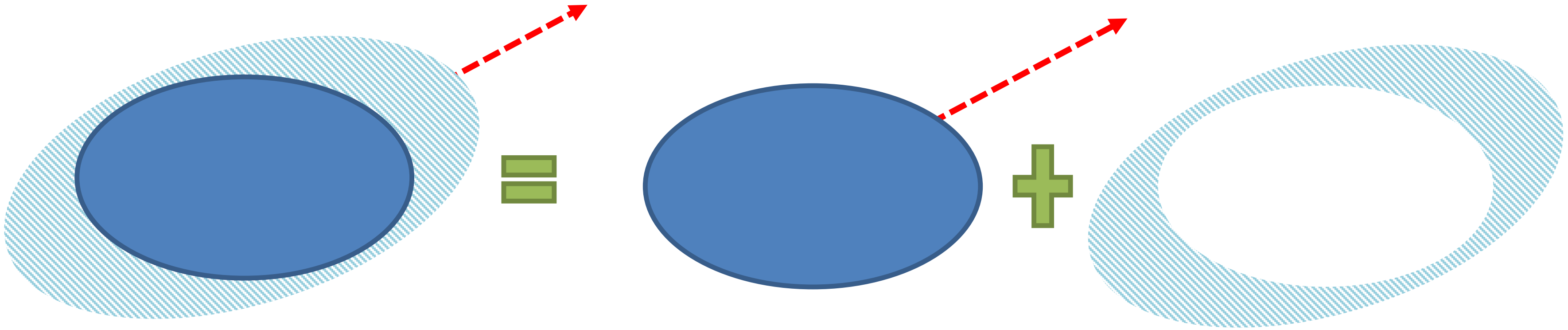
Context and classical formulae

- From a theoretical perspective, the (tidal) redistribution effects on the Earth rotation can be modelled through the next sequence:
  1. External gravitational interactions create a potential on the Earth, which is deformed under these (and other) actions
  2. That causes a redistribution of mass within the Earth system
  3. That redistribution originates time variations of the gravitational energy that are treated as an additional potential superposed on a theoretical “unperturbed" potential (tidal redistribution potential)
  4. In turn, the tidal redistribution potential affects Earth rotation



Actual Earth = ideal Earth in undeformed state + deformations

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Context and classical formulae

- A simple way to implement such sequence is via the Earth matrix of inertia

$$I'_{ij}=\int_{B'}\rho'\left(\mathbf{r}'\right)\left(r'^2\delta_{ij}-x'_ix'_j\right)d^3\mathbf{r}',$$

- That matrix is linked with the gravitational potential energy (2nd degree term) by the MacCullagh's formula (1844)

$$\mathcal{V}'\left(\mathbf{r}_p\right)=-\frac{Gm_p}{2r_p^5}\sum_{i,j=1(j\geq i)}^{3}I'_{ij}\left[r_p^2\delta_{ij}-\left(3-2\delta_{ij}\right)x_{i,p}x_{j,p}\right],$$

with $\mathbf{r}_p=(x_{1,p},x_{2,p},x_{3,p})$ the perturber position and $r_p=|\mathbf{r}_p|$

- The mass redistribution is isolated in the inertia matrix by solving the elastic problem for the tide-raising potential $\mathcal{U}_2(\mathbf{r},\mathbf{r}_q)$ caused by the perturbing body $\mathbf{r}_q$

$$\mathcal{U}_2\left(\mathbf{r},\mathbf{r}_q\right)=-G\frac{m_q}{r_q}\left(\frac{r}{r_q}\right)^2P_2\left(\cos\gamma\right),\qquad \mathbf{r}'=\mathbf{r}+\mathbf{u}(\mathbf{r}),\ \rho'(\mathbf{r}')=\rho(\mathbf{r})-\rho(\mathbf{r})\nabla\cdot\mathbf{u}(\mathbf{r})$$

- For example, for a SNREI (Spherical, Non-Rotating, Elastic, Isotropic) Earth we have

$$\mathbf{u}(\mathbf{r})=-F_2\left(r\right)\nabla\mathcal{U}_2\left(\mathbf{r},\mathbf{r}_q\right)-G_2\left(r\right)\mathcal{U}_2\left(\mathbf{r},\mathbf{r}_q\right)\mathbf{r}$$

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Context and classical formulae

- In this way, the matrix of inertia is split in "undeformed" and deformed parts
- The "undeformed" part is related to a "fictitious" Earth characterized by the matrix of inertia $\mathbf{I} = \text{diag}\,\{\bar{A}, \bar{A}, C\}$ —constant in a certain body axes system
- Introducing the solid spherical harmonics of the 2nd degree (omitting θ_p, ϕ_p)

$$r_p^2 \mathcal{C}_{20,p} = \frac{1}{2}\left(3x_{3,p}^2 - r_p^2\right), \quad \begin{aligned} r_p^2 \mathcal{C}_{21,p} &= 3x_{1,p}x_{3,p}, \\ r_p^2 \mathcal{S}_{21,p} &= 3x_{2,p}x_{3,p}, \end{aligned} \quad \begin{aligned} r_p^2 \mathcal{C}_{22,p} &= 3\left(x_{1,p}^2 - x_{2,p}^2\right), \\ r_p^2 \mathcal{S}_{22,p} &= 6x_{1,p}x_{2,p}, \end{aligned}$$

the deformed part associated to the redistribution of mass is

$$\Delta\mathbf{I}_q(t) = k_2\left(\frac{m_q a_E^5}{3r_p^3}\right)\begin{pmatrix} \mathcal{C}_{20,q} - \frac{1}{2}\mathcal{C}_{22,q} & -\frac{1}{2}\mathcal{S}_{22,q} & -\mathcal{C}_{21,q} \\ -\frac{1}{2}\mathcal{S}_{22,q} & \mathcal{C}_{20,q} + \frac{1}{2}\mathcal{C}_{22,q} & -\mathcal{S}_{21,q} \\ -\mathcal{C}_{21,q} & -\mathcal{S}_{21,q} & -2\mathcal{C}_{20,q} \end{pmatrix}, \quad \underbrace{<\Delta\mathbf{I}_q(t)> \neq \mathbf{0}}_{(\text{ permanent tide})}$$

- In the previous formula, all the elastic response of the Earth is described in the constant parameter k_2 (Love number)

$$k_2 = \frac{8\pi}{5}\frac{G}{a_E^5}\int_0^{a_E} r^4\rho(r)\left[5F_2(r) + r^2G_2(r)\right]dr$$

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Context and classical formulae

- From the expressions of the "undeformed" and deformed parts of the matrix of inertia, one can compute the corresponding ones of the gravitational potential energy
- The undeformed part **I** leads to the "ordinary" potential energy (J_2 term)

$$\mathcal{V}\left(\mathbf{r}_p\right)=\frac{Gm_p}{2r_p^5}(C-\bar{A})\left(3x_{3,p}-r_p^2\right)=\frac{Gm_p}{r_p^3}(C-\bar{A})\mathcal{C}_{20,p}(\theta_p,\phi_p)=\frac{Gm_p}{r_p^3}(C-\bar{A})\mathcal{C}_{20,p}$$

- The deformed part (redistribution of the elastic Earth caused by the direct action of the perturbers) gives the redistribution tidal potential expression

$$\begin{aligned}\mathcal{V}_{t;p,q}\left(\mathbf{r}_p\right) &= -Ga_E^5\frac{m_pm_q}{r_p^3r_q^3}k_2\times\Big[\mathcal{C}_{20,p}\mathcal{C}_{20,q}+\frac{1}{3}\left(\mathcal{C}_{21,p}\mathcal{C}_{21,q}+\mathcal{S}_{21,p}\mathcal{S}_{21,q}\right)+\\ &+\frac{1}{12}\left(\mathcal{C}_{22,p}\mathcal{C}_{22,q}+\mathcal{S}_{22,p}\mathcal{S}_{22,q}\right)\Big]\end{aligned}$$

- It is important to distinguish the roles of the perturbers both as perturbed $\mathbf{r}_p$ and perturbing bodies $\mathbf{r}_q$ (given functions of t, in case of confusion we will add a tilde ~)

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Context and classical formulae

- In the previous case —SNREI Earth model—, the redistribution tidal potential is proportional to the tide-raising potential

$$\frac{\mathcal{V}_{t;p,q}(\mathbf{r}_p)}{m_p} = k_2 \left(\frac{a_E}{r_p}\right)^5 \mathcal{U}_2(\mathbf{r}_p, \mathbf{r}_q),$$

  which is the usual way to introduce the Love number

- However, more realistic Earth models change the nature of k_2, which transforms into a function of time, i.e., into Love functions
- For example, anelastic features (e.g., Efroimsky 2012) entail a time lag between the cause and the effect

$$\frac{\mathcal{V}_{t;p,q}(t)}{m_p} = \left(\frac{a_E}{r_p}\right)^3 \int_{-\infty}^{t} \dot{k}_2(t-t')\,\mathcal{U}_2(\mathbf{r}_q, t')\,dt'$$

- It is easier tackled with the frequency-dependent complex Love functions $\bar{k}_2(\omega)$

$$\frac{\bar{\mathcal{V}}_{t;p,q}(\omega)}{m_p} = \bar{k}_2(\omega) \left(\frac{a_E}{r_p}\right)^3 \bar{\mathcal{U}}_2(\mathbf{r}_q, \omega), \qquad \bar{k}_2(\omega) = \left|\bar{k}_2(\omega)\right| e^{i\varepsilon(\omega)}$$

  leading to need of considering a harmonic decomposition of the motion of the perturbers (tidal constituents)

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Context and classical formulae

- In a similar way, the influence of the Earth rotation and ellipticity on tides also make the Love numbers frequency dependent (e.g., Smith 1974)
- The corresponding expressions provide expansions (real or complex) in the frequency domain of the form

$$\bar{k}_2(\omega) = L_0 + \sum_i \frac{L_i}{\omega - \omega_i},$$

  where ω_i denotes the normal mode (e.g., CW, FCN, etc.) of the corresponding rotation Earth model —rotation couples tides
- In addition to the features already mentioned, related to the solid Earth —anelasticity, ellipticity and rotation—, there is also a significant contribution from oceans, so we can write

$$\bar{k}_2(\omega) = \left[\bar{k}_{2\,el}(\omega) + \bar{k}_{2\,an}(\omega) + \bar{k}_{2\,in}(\omega)\right] + \bar{k}_{2\,oc}(\omega),$$

- The oceanic part $\bar{k}_{oc}(\omega)$ is the main responsible of phase lag $\varepsilon(\omega)$ and induces indirect contributions (loads) in the solid Earth $\bar{k}_{in}(\omega)$

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Context and classical formulae

- In the former expressions the frequency of the tide-raising potential ω can be obtained from the orbital ephemeris of the perturbers as

$$\omega = m\omega_E \pm n_j,\ m = 0, 1, 2,\ j \in \mathcal{F}$$

where n_j ($|n_j| < \omega_E$) is its frequency relative to a quasi-inertial reference system

- Such decomposition lead to group the frequencies in bands (*m*) and then variations across each band, hence we also employ the notation $\bar{k}_{2m,j} = |\bar{k}_{2m,j}|\mathrm{e}^{i\varepsilon_{2m,j}}$
- The particular frequency dependence rests on the Earth rheology assumed and is difficult to model, some of the available Love number sets are:
  - Solid Earth:
    - Frequency independent (SNREI Earth): linear elastic —cancellation, permanent tide
    - Frequency dependence per band (IERS Conventions 2010): anelas., rot. and ellip.
    - Frequency dependence (IERS Conventions 2010): anelas., rot., ellip., and ocean load
  - Ocean:
    - Frequency dependence (Willams & Boggs 2016): from FES2004 (Lyard et al. 2006)

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Context and classical formulae

- Hence, from the SNREI case redistribution tidal potential of the

$$\mathcal{V}_{t;p,q}(\mathbf{r}_p) = -\,G a_E^5 \frac{m_p m_q}{r_p^3 r_q^3} k_2 \times \left[\mathcal{C}_{20,p}\mathcal{C}_{20,q} + \frac{1}{3}\left(\mathcal{C}_{21,p}\mathcal{C}_{21,q} + \mathcal{S}_{21,p}\mathcal{S}_{21,q}\right) + \frac{1}{12}\left(\mathcal{C}_{22,p}\mathcal{C}_{22,q} + \mathcal{S}_{22,p}\mathcal{S}_{22,q}\right)\right],$$

we move to an Earth model with Love functions $\bar{k}(\omega) = \bar{k}_{2m,j} = |\bar{k}_{2m,j}|e^{i\varepsilon_{2m,j}}$

$$\begin{aligned}\mathcal{V}_{t;p,q}(\mathbf{r}_p) &= -G a_E^5 \sum_j \frac{m_p m_q}{r_q^3 r_p^3} \times \Bigg\{ \underbrace{|\bar{k}_{20,j}| \cos\varepsilon_{20} \mathcal{C}_{20,p}(\theta_p,\phi_p)\,\mathcal{C}_{20,q}(\theta_q,\phi_q)}_{\text{zonal}} + \\ &+ \underbrace{\frac{1}{3}|\bar{k}_{21,j}|\left[\mathcal{C}_{21,p}(\theta_p,\phi_p)\,\mathcal{C}_{21,q}(\theta_q,\phi_q-\varepsilon_{21,j}) + \mathcal{S}_{21,p}(\theta_p,\phi_p)\,\mathcal{S}_{21,q}(\theta_q,\phi_q-\varepsilon_{21,j})\right]}_{\text{diurnal}} + \\ &+ \underbrace{\frac{1}{12}|\bar{k}_{22,j}|\left[\mathcal{C}_{22,p}(\theta_p,\phi_p)\,\mathcal{C}_{22,q}\left(\theta_q,\phi_q-\frac{\varepsilon_{22,j}}{2}\right) + \mathcal{S}_{22,p}(\theta_p,\phi_p)\,\mathcal{S}_{22,q}\left(\theta_q,\phi_q-\frac{\varepsilon_{22,j}}{2}\right)\right]}_{\text{semidiurnal}} \Bigg\}.\end{aligned}$$

- It is implicitly assumed that the real spherical harmonics for perturbed and perturbing bodies are also given as harmonic sums in their orbital frequencies

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Context and classical formulae

- The contributions of the redistribution tidal potential to the Earth rotation have also been computed in other works, with different scopes, like in:

Authors	Redistr.	Earth model	Analyt.	Topic
Getino & Ferrándiz (1991)	AE	1L	Yes, H	LOD
Krasinsky (1999, 2003, 2006)	AE	1L, 2L	Yes (2006, No)	P-N-LOD
Ray et al. (1999)	OT	1L (atm.)	No	LOD
Souchay & Folgueira (2000)	E	1L (partial)	Yes, H	N
Escapa et al. (2003, 2004)	E, AE	2L	Yes, H	P-N
Mathews et al. (2002, 2003)	AE	2L (2002, partial)	No	P-N
Lambert & Capitaine (2004)	AE	2L (partial)	No	P-N
Lambert & Mathews (2006, 2008) Mathews & Lambert (2009)	AE+OT	2L	No	P-N-LOD
Williams & Boggs (2016)	AE+OT	2L	Yes	P-LOD
Baenas et al. (2019, 2020, 2021)	**AE+OT**	**2L**	**Yes, H**	**P-N-LOD**

E: elastic; AE: anelasticity of the mantle; OT: oceanic tide; 1L/2L: one-layer/two-layer Earth model
P-N-LOD: precession, nutation, or length of day; H: Hamiltonian

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Context and classical formulae

- Lambert & Mathews (2006, 2008 erratum) and Mathews & Lambert (2009) considered a two-layer Earth model, deriving the effects of the redistribution for precession, nutation, and LOD secular variations from the computation of its torque in SOS (Sasao et al. 1980) framework
- They took Love number sets including frequency dependence for the solid Earth and the direct contributions for oceans constructed from CSR4.0 (Eanes, 2002)
- Their results are numerical and some details were not explicitly stated (e.g., the treatment of the permanent tide
- Williams & Boggs (2016) considered quite similar Earth model than in the previous case, with a Love number set for the solid Earth from IERS Conventions 2010 and that for the oceans constructed from FES2004 (Lyard et al. 2006, recommended by IERS)
- They gave the secular variations of LOD and the obliquity (lunar orbit) from the torque equations with analytical formulae, deriving the redistribution effects from its potential
- There are discrepancies in the common obtained results: e.g., the obliquity rate values are 1.84 cy^{-1} (Lambert & Mathews 2008) and 0.92 cy^{-1} (Williams & Boggs 2016)

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Context and classical formulae

- Having in mind the described situation, we undertook the task of developing a new model for the contributions of the tidal redistribution of mass to the Earth rotation
- It has been partially accomplished in a set of papers by Baenas, Escapa, & Ferrándiz (2019, 2020, 2021)
- They main features that we required to such model were to be:
  - Consistent both from an internal perspective and also with other parts of the Earth rotation theory (e.g., ordinary precession and nutation)
  - Analytical in order to make easy the numerical computations with different standards (IERS Conventions update) and the comparisons with similar researches
  - Ready, or almost ready, to use with different Love number sets

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Methods

- The former requirements can be achieved by developing the tidal mass redistribution effects within the Hamiltonian theory of the rotation of the non-rigid Earth

- The rotational motion is derived from Hamilton's equations

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{Q}_{q_i}, \ \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{Q}_{p_i}, \ i = 1, \ldots, n$$

  - Canonical set: (p, q); Hamiltonian: $\mathcal{H}$; Canonical, or generalized, forces: $\mathcal{Q}_p, \mathcal{Q}_q$

- It allows to obtain the time evolution of any function of the phase space $f(p, q)$,

$$\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t} - \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} Q_{p_i} - \frac{\partial f}{\partial p_i} Q_{q_i} \right)$$

  where $\{-, -\}$ denotes de Poisson bracket of two functions in the set (p, q)

- Since the direct resolution of the Hamiltonian equations is unfeasible, one can employ analytical perturbation methods that are available in the Hamiltonian framework (e.g., Hori 1966, 1971; Baenas et al. 2017, etc.)

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Methods

- The baseline Earth model for the computations will be a two-layer Earth model composed of a non-rigid mantle and a fluid core with a dissipative effects at the core mantle boundary (CMB)
- The Hamiltonian describing the rotation of such model is

$$\mathcal{H} = \mathcal{T}+\mathcal{V}_0+\mathcal{E}+\mathcal{T}_t+\mathcal{V}_t \Rightarrow \begin{cases} \mathcal{T}: \text{ kinetic energy} \\ \mathcal{V}_0: \; J_2 \text{ 'unperturbed' part of Earth geopotential} \\ \mathcal{E}: \text{motion of ecliptic of date} \\ \mathcal{T}_t: \text{tidal redistribution kinetic energy} \\ \mathcal{V}_t: \text{tidal redistribution potential energy} \end{cases}$$

- As in former studies, we use an Andoyer-like canonical variables (12 dim. space phase) to formulate the problem (Getino 1995)

$$M, N, \Lambda; \mu, \nu, \lambda \longrightarrow \text{Whole Earth}$$
$$M_c, N_c, \Lambda_c; \mu_c, \nu_c, \lambda_c \longrightarrow \text{Fluid core}$$

$$R = R_3(\nu)\, R_1(\sigma)\, R_3(\mu)\, R_1(I)\, R_3(\lambda)$$

$$\Lambda = M\cos I, \; N = M\cos\sigma$$
$$N = M\cos\sigma, \; N_c = M_c\cos\sigma_c$$

$$\vec{L} = \begin{pmatrix} M\sin\sigma\sin\nu \\ M\sin\sigma\cos\nu \\ N \end{pmatrix}, \; \vec{L}_c = \begin{pmatrix} M_c\sin\sigma_c\sin\nu_c \\ -M_c\sin\sigma_c\cos\nu_c \\ N_c \end{pmatrix}$$

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Methods

- With these elements, the general first order scheme of Hori's method plus averaging is implemented —for simplicity in the explanations the conservative version is just considered here— through:
  - The original Hamiltonian

$$\mathcal{H} = \mathcal{T} + \overbrace{\mathcal{V}_0 + \mathcal{E} + \mathcal{T}_t + \mathcal{V}_t}^{\mathcal{H}_1} = \mathcal{H}_0 + \mathcal{H}_1 + \ldots$$

  - The generating function and the transformed Hamiltonian

$$\mathcal{W} = \mathcal{W}_1 = \int_{UP} \mathcal{H}_{1\ \mathrm{per}}\, dt \qquad \begin{aligned} \mathcal{H}^* &= \mathcal{H}_0^* + \mathcal{H}_1^*, \\ \mathcal{H}_0^* &= \mathcal{H}_0,\ \mathcal{H}_1^* = \mathcal{H}_{1\ \mathrm{sec}} \end{aligned}$$

  - The averages and integrals are computed over the unperturbed problem *UP* — main part of $\mathcal{T}$—, which involves the CW and FCN modes
- By doing so, the time variation of any function *f* of the canonical variables is given by

$$f = f^* + \{f^*; \mathcal{W}_1\} \longrightarrow \left\{ \begin{array}{lll} \Delta f = & \{f^*; \mathcal{W}_1\} & \text{(periodic part)} \\ \dfrac{df}{dt} = & \{f^*; \mathcal{H}^*\} & \text{(secular part)} \end{array} \right.$$

- So, at the first order, the effects of the perturbations can be tackled independently

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Methods

- Therefore, the contribution of the tidal mass redistribution potential on any function $f(p,q)$ of the rotational dynamics of the non-rigid Earth are given by

$$\delta\left(\frac{df}{dt}\right) = \delta n_f = \{f^*; \mathcal{V}_{t,\,\mathrm{sec}}\} \quad \rightarrow \quad \text{Contribution to the rate of } f$$

$$\Delta f = \left\{f^*; \int_{UP} \mathcal{V}_{t,\,\mathrm{per}}\, dt\right\} \quad \rightarrow \quad \text{Contribution to the periodic part of } f$$

- The functions to be considered here are:
  - The longitude λ_f and obliquity I_f of the figure axis (opposite sign to the conventional one) —secular contributions and periodic contributions

$$\begin{array}{l} \lambda_f = \lambda + \sigma \dfrac{\sin\mu}{\sin I}, \\ I_f = I + \sigma\cos\mu \end{array} \longrightarrow \left\{ \begin{array}{lll} \delta n_{\lambda_f} = \delta n_\lambda; & \delta n_{I_f} = \delta n_I & \text{(precession rates)} \\ \Delta\lambda_f = \Delta\lambda + \Delta\left(\sigma\dfrac{\sin\mu}{\sin I}\right); & \Delta I_f = \Delta I + \Delta(\sigma\cos\mu) & \text{(nutations)} \end{array} \right.$$

  - The third component of the angular velocity in the terrestrial system —secular contribution

$$\omega_z = \frac{1}{C_m}(N - N_c) \longrightarrow -\frac{\overline{\mathrm{LOD}}}{\omega_E}\delta\left(\frac{d\omega_z}{dt}\right) = \delta\left(\frac{d\mathrm{LOD}}{dt}\right) \quad \text{(LOD rate)}$$

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Methods

- The remaining process, which allows the derivation of analytical formulae for the considered quantities, is carried out by:
  - Expressing the redistribution tidal potential in terms of the canonical variables
  - Separating its secular part (rates)
  - Separating its periodic part and computing its generating functions (nutations)
- So, the central point is to write out

$$
\begin{aligned}
\mathcal{V}_{t;p,q}\left(\mathbf{r}_p\right) &= -G a_E^5 \sum_j \frac{m_p m_q}{r_q^3 r_p^3} \times \Bigg\{ \underbrace{\left|\bar{k}_{20,j}\right| \cos \varepsilon_{20} \mathcal{C}_{20,p}\left(\theta_p, \phi_p\right) \mathcal{C}_{20,q}\left(\theta_q, \phi_q\right)}_{\text{zonal}} + \\
&\underbrace{+ \frac{1}{3}\left|\bar{k}_{21,j}\right| \left[\mathcal{C}_{21,p}\left(\theta_p, \phi_p\right) \mathcal{C}_{21,q}\left(\theta_q, \phi_q - \varepsilon_{21,j}\right) + \mathcal{S}_{21,p}\left(\theta_p, \phi_p\right) \mathcal{S}_{21,q}\left(\theta_q, \phi_q - \varepsilon_{21,j}\right)\right]}_{\text{diurnal}} + \\
&\underbrace{+ \frac{1}{12}\left|\bar{k}_{22,j}\right| \left[\mathcal{C}_{22,p}\left(\theta_p, \phi_p\right) \mathcal{C}_{22,q}\left(\theta_q, \phi_q - \frac{\varepsilon_{22,j}}{2}\right) + \mathcal{S}_{22,p}\left(\theta_p, \phi_p\right) \mathcal{S}_{22,q}\left(\theta_q, \phi_q - \frac{\varepsilon_{22,j}}{2}\right)\right]}_{\text{semidiurnal}} \Bigg\} .
\end{aligned}
$$

as a function of the canonical Andoyer-like variables

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Methods

- The coordinates of the perturbers, both as perturbed and perturbing bodies, enter through the second degree solid spherical harmonics with respect to the terrestrial system
- However, their evolution is known in the ecliptic of date by some ephemeris like, in our case, ELP for the Moon and VSOP for the Sun
- Within the Hamiltonian framework such transformation involves the Andoyer variables —describing the rotation of the Earth— and some orbital functions and variables that characterize the Moon and the Sun motions (Kinoshita 1977, Getino & Ferrándiz 1995). As an example, we have

$$\begin{aligned}
\left(\frac{a}{r}\right)^3 \mathcal{C}_{20} &= 3\sum_i B_i(I)\cos\Theta_i - 3\sigma \sum_{i,\tau=\pm1} C_i(I,\tau)\cos(\mu-\tau\Theta_i)\,, \\
\left(\frac{a}{r}\right)^3 \mathcal{C}_{21} &= 3\sum_{i,\tau=\pm1} C_i(I,\tau)\sin(\mu+\nu-\tau\Theta_i) + \sigma\sum_{i,\tau=\pm1}\left[\frac{9}{2}B_i(I)\sin(\nu-\tau\Theta_i) - \frac{3}{2}D_i(I,\tau)\sin(2\mu+\nu-\tau\Theta_i)\right], \\
\left(\frac{a}{r}\right)^3 \mathcal{C}_{22} &= -3\sum_{i,\tau=\pm1} D_i(I,\tau)\cos(2\mu+2\nu-\tau\Theta_i) - 6\sigma\sum_{i,\tau=\pm1} C_i(I,\tau)\cos(\mu+2\nu-\tau\Theta_i)
\end{aligned}$$

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- In the former expressions, the orbital motion of the Moon and the Sun is characterized by:
  - A combination of the Delaunay variables of the Moon and the Sun

$$\Theta_i = m_{i1} l_M + m_{i2} l_S + m_{i3} F + m_{i4} D + m_{i5} \left(\Omega - \lambda \right) = n_i t + \Theta_{i0}$$

  - A quintuplet of integers, running on the set of orbital frequencies

$$i = \left(m_{i1}, m_{i2}, m_{i3}, m_{i4}, m_{i5} \right), \; i \in \mathcal{F},$$

    which includes the case *(0,0,0,0,0)*

  - Kinoshita (1977) orbital functions $B_i(I)$, $C_i(I,\tau)$, $D_i(I,\tau)$, e.g., $B_i(I)$ is

$$B_i(I) = -\frac{1}{6} \left(3\cos^2 I - 1 \right) A_i^{(0)} - \frac{1}{2} \sin 2I A_i^{(1)} - \frac{1}{4} \sin^2 I A_i^{(2)}$$

- When considering the perturbing bodies (known functions of *t*), the previous expansions can be simplified to

$$\lambda = \lambda_0, \; M = C\omega_E, \; I = I_0, \; \sigma = 0, \; \mu + \nu = \omega_E t + (\mu + \nu)_0$$

  and, to avoid confusions, the tilde notation be introduced

$$\tilde{I}, \; \tilde{\mu}, \; \tilde{\nu}, \; \tilde{B}_{j;q}, \; \text{etc.}$$

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Methods

- For example, the part of the redistribution tidal potential in σ^0 for each band m reads as

$$\mathcal{V}_{t,0}^{(0)} = -\frac{9}{4} C\omega_E \sum_{p,q,i,j,\tau,\epsilon} f_q k_p \left|\bar{k}_{20,j}\right| B_{i;p} \tilde{B}_{j;q} \cos\left(\tau\Theta_i - \epsilon\tilde{\Theta}_j - \varepsilon_{20,j}\right),$$

$$\mathcal{V}_{t,0}^{(1)} = -3C\omega_E \sum_{p,q,i,j,\tau,\epsilon} f_q k_p \left|\bar{k}_{21,j}\right| C_{i;p} \tilde{C}_{j;q} \cos\left(\mu + \nu - \tau\Theta_i - \tilde{\mu} - \tilde{\nu} + \epsilon\tilde{\Theta}_j + \varepsilon_{21,j}\right),$$

$$\mathcal{V}_{t,0}^{(2)} = -\frac{3}{4} C\omega_E \sum_{p,q,i,j,\tau,\epsilon} f_q k_p \left|\bar{k}_{22,j}\right| D_{i;p} \tilde{D}_{j;q} \cos\left(2\mu + 2\nu - \tau\Theta_i - 2\tilde{\mu} - 2\tilde{\nu} + \epsilon\tilde{\Theta}_j + \varepsilon_{22,j}\right)$$

with the parameters related to the Earth model

$$k_p = \frac{3Gm_p}{\omega_E a_p^3} H_d, \; f_q = \frac{m_q a_E^2}{3CH_d}\left(\frac{a_E}{a_q}\right)^3$$

- From it, one can separate the secular part considering

$$\mathcal{I} = \left\{\tau, \epsilon \in \{-1, +1\} \mid \tau\Theta_i - \epsilon\tilde{\Theta}_j = 0\right\}$$

- A similar expression arises for the part proportional to σ^1, which just contributes to the nutations (Oppolzer terms)

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Results

- The described procedure allows obtaining the analytical contributions to precession, nutation, and LOD secular change from the tidal mass redistribution
- For example, the resulting formulae for the precession rates are

$$\delta n_\psi = -\delta n_\lambda = -\frac{1}{\sin I}\sum_{p,q=M,S} f_q k_p \sum_{\substack{i,j;\tau,\epsilon\in\mathcal{I}\\ m=0,1,2}} \left|\bar{k}_{2m,j}\right| T^{(n_\psi)}_{ijpq,m}\cos\varepsilon_{2m,j},$$

$$\delta n_\varepsilon = -\delta n_I = -\frac{1}{\sin I}\sum_{p,q=M,S} f_q k_p \sum_{\substack{i,j;\tau,\epsilon\in\mathcal{I}\\ m=0,1,2}} \left|\bar{k}_{2m,j}\right| T^{(n_\varepsilon)}_{ijpq,m}\sin\varepsilon_{2m,j},$$

$$T^{(n_\psi)}_{ijpq,m} = \frac{9}{4}\frac{\partial B_{i;p}}{\partial I}\tilde{B}_{j;q}\delta_{m0} + 3\frac{\partial C_{i;p}}{\partial I}\tilde{C}_{j;q}\delta_{m1} + \frac{3}{4}\frac{\partial D_{i;p}}{\partial I}\tilde{D}_{j;q}\delta_{m2},$$

$$T^{(n_\varepsilon)}_{ijpq,m} = -\frac{9}{4}B_{i;p}\tilde{B}_{j;q}\tau m_{5i}\delta_{m0} + 3C_{i;p}\tilde{C}_{j;q}\left(\tau m_{5i} - \cos I\right)\delta_{m1} + \frac{3}{4}D_{i;p}\tilde{D}_{j;q}\left(\tau m_{5i} - 2\cos I\right)\delta_{m2}$$

showing neatly the dependences on the orbital functions, the Love numbers, and the Earth model parameters

- Similar expressions were derived for the nutations and the secular change of LOD

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Results

- The numerical evaluation of our precession formulae lead to (unit: mas cy^{-1})

	Elastic linear	Anelastic IERS k_{2m}	Anelastic IERS $k_{2m;j}$	Anelastic WB2016 $k_{2m;j}$
Longitude rate				
Zonal permanent tide with k_{2f}	136.5964	136.5964	136.5964	136.5964
Zonal permanent tide with k_2	43.5946	43.5946	43.5946	43.5946
non-permanent (Znp)	−4.1064	−4.1324	−4.1484	−4.5780
Tesseral (T)	−66.4701	−66.0934	−60.7723	−64.6514
Sectorial (S)	26.9818	27.0736	27.0736	25.2844
Total (Znp+T+S)	**−43.5946**	**−43.1522**	**−37.8472**	**−43.9450**
(including permanent tide with k_{2f})	(93.0018)	(93.4442)	(98.7492)	(92.6514)
(including permanent tide with k_2)	(0.0000)	(0.4424)	(5.7474)	(−0.3504)
Obliquity rate				
Zonal non-permanent (Znp)	0.0000	0.0000	−0.0118	−0.0636
Tesseral (T)	0.0000	−0.0239	0.0404	0.0948
Sectorial (S)	0.0000	0.0465	0.0465	0.9030
Total (Znp+T+S)	**0.0000**	**0.0226**	**0.0751**	**0.9341**

- Cancellation for the SNREI model when including the permanent tide ($k_{2f} = k_2$)
- Very good agreement with Williams & Boggs' (2016) obliquity rate (0.92 mas cy^{-1})

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Results

- As a byproduct of our developments, we also highlighted the role of the permanent tide both in computing the redistribution tidal contributions and in other parameters derived from the Earth rotation (e.g., the dynamical ellipticity *H*)
- In particular, the time average value of the deformed part of the matrix of inertia is

$$< \Delta \mathbf{I}_q(t) >= k_2 \left(\frac{m_q a_\oplus^5}{a_q^3} \right) B_{0,q}(I) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- From this expression it is possible to obtain (Escapa el al. 2020) the contribution of the zero tidal distortion of *H* —Darwin's theorem is involved

$$\delta H \simeq \frac{3}{2} \frac{\Delta C}{C} = -k_2 \left(\frac{3 a_\oplus^5}{C} \right) \sum_{q=m,s} \left(\frac{m_q}{a_q^3} \right) B_{0,q} \simeq 8.8716 \times 10^{-8} k_2$$

- In this regard, it is important to recall that the redistribution contributions to the precession and nutation must not contain the permanent tide, since that part is accounted for in the ordinary precession and nutation —it avoids giving a value to k_{2f} as recommended by IAG (zero-frequency system)

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Results

- The numerical evaluation of our nutation formulae lead to the following values (unit: micro as) for the Love number set given by the IERS Conventions 2010 (frequency dependent) and Williams & Boggs (2016)

	Poisson (subscript 1) and Oppolzer (subscript 2) terms								Total			
Period	Longitude				Obliquity				Longitude		Obliquity	
(days)	$\mathcal{L}^{in,1}$	$\mathcal{L}^{in,2}$	$\mathcal{L}^{out,1}$	$\mathcal{L}^{out,2}$	$\mathcal{O}^{in,1}$	$\mathcal{O}^{in,2}$	$\mathcal{O}^{out,1}$	$\mathcal{O}^{out,1}$	$\mathcal{L}^{in}$	$\mathcal{L}^{out}$	$\mathcal{O}^{in}$	$\mathcal{O}^{out}$
−6798.38	199.02	2.09	10.70	−0.09	−95.34	−0.71	6.29	−0.03	**201.11**	**10.60**	**−96.04**	**6.26**
−3399.19	−1.71	−0.03	0.29	0.00	0.71	0.01	0.10	0.00	**−1.74**	**0.29**	**0.72**	**0.10**
365.26	−1.02	0.48	0.03	−0.01	−0.01	0.16	0.02	0.01	**−0.54**	**0.02**	**0.15**	**0.03**
182.62	11.30	−1.05	0.91	0.09	−4.71	0.47	0.36	0.03	**10.25**	**1.01**	**−4.25**	**0.39**
27.55	−0.65	0.08	−0.03	0.00	0.00	−0.01	−0.05	−0.01	**−0.57**	**−0.03**	**−0.01**	**−0.06**
13.66	1.92	−0.42	0.14	0.04	−0.81	0.17	0.06	0.01	**1.51**	**0.18**	**−0.64**	**0.07**

- From our analytical expression it is also possible to compute the results for the other Love number sets, we also found exact cancellation for the SNREI model when including the permanent tide ($k_{2f} = k_2$)
- As in the precessional case, the discrepancies with Lambert & Mathews (2006, 2008 erratum) were evident

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Results

- The numerical evaluation of our LOD change formulae lead to the following values (coupled core-mantle) for the Love number set given by the IERS Conventions 2010 (frequency dependent) and Williams & Boggs (2016)

Potential component		Solid tides (IERS2010)	Ocean tides (WB2016)	Total
Tesseral	$('' /\mathrm{cy}^2)$	2.4	-193.5	-191.1
$(m = 1)$	(ms/cy)	-0.004	0.352	0.348
Sectorial	$('' /\mathrm{cy}^2)$	-61.7	-1075.8	-1137.5
$(m = 2)$	(ms/cy)	0.112	1.9589	2.071
Total	$('' /\mathrm{cy}^2)$	-59.3	-1269.3	-1328.6
Two–layer Earth	(ms/cy)	0.108	2.310	**2.418**

- Our analytical treatment also allows to compute the LOD change for different degrees of coupling between the mantle and the core in their secular evolution (due to the dissipative torque at the CMB)
- Very good agreement with Williams & Boggs' (2016) value (2.50 ms cy^{-1}) —Mathew & Lambert 2009 provide 2.40 ms cy^{-1}

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Summary

- We have derived a new model for the contributions to the Earth rotation of the redistribution potential due to the tidal redistribution of mass
- It provides closed-analytical formulae derived within the Hamiltonian formalism, particularly to precession, nutation, and secular change of LOD
- It considers a two-layer Earth, with dissipation at the CMB; the tidal mass redistribution characterized for two sets of Love numbers related to the solid and oceanic tides
- In particular for the Love numbers sets given by IERS Conventions 2010 —solid tides— and William & Boggs (2016) —oceanic tides, we obtained:
  - A precession rate in longitude and obliquity of -43.95 mas cy^{-1} and 0.9341 mas cy^{-1}
  - Nutation amplitudes greater than 1 μas for 6 terms, with the leading one (μas)

    $$\Delta\psi = 201.11 \sin\Omega + 10.60 \cos\Omega, \Delta\varepsilon = -96.04 \cos\Omega + 6.26 \sin\Omega$$

  - A LOD secular change of 2.42 ms cy^{-1}
- Our developments provide a clear treatment of the permanent tide, lead to an exact cancellation for a SNREI Earth model, and give very good agreement with the values derived in William & Boggs (2016) for the obliquity rate and the secular change of LOD

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Summary

- They main advantages of our approach, based on the Hamiltonian theory of the non-rigid Earth, are that it is:
  - Consistent both from an internal perspective and also with other parts of the Earth rotation theory (e.g., ordinary precession and nutation), since it is derived from the same Hamiltonian
  - Analytical, so it makes easy the numerical computations with different standards (IERS Conventions update) and the comparisons with similar researches
  - Ready, or almost ready, to use with different Earth rheological and oceanic models (Love number sets)

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